Journal of Engineering Mathematics (2005) 52: 53–61
DOI 10.1007/s10665-005-0442-8

Some theoretical results about second-order work, uniqueness, existence and controllability independent of the constitutive equation

RENÉ CHAMBON
Laboratoire 3S Université Joseph Fourier, Institut National Polytechnique, CNRS, UMR 5521, B.P. 53 38041 Grenoble Cedex France (rene.chambon@inpg.fr)

Received 6 April 2004; accepted in revised form 6 January 2005

Abstract. When they are studied as continuum media, granular materials and other soils and rocks exhibit a complex behavior. Contrary to metals, their isotropic and deviatoric behavior are coupled. This implies some mathematical difficulties concerning boundary-value problems solved with constitutive equations modelling the salient features of such geomaterials. One of the well-known consequences is that the so-called second-order work can be negative long before theoretical failure occurs. Keeping this in mind, the starting point of this work is the pioneering and illuminating work of Nova (1994), who proved that using an isotropic hardening elasto-plastic model not obeying the normality rule, it is possible to exhibit either loss of uniqueness or loss of existence of the solution of a boundary-value problem as soon as the second-order work is negative. Because the geomaterial behavior is quite difficult to model, in practice many different constitutive equations are used. It is then important to study the point raised by Nova for other constitutive equations. In this paper, his result is generalized for any inelastic rate-independent constitutive equation. Similarly the link between localization and controllability proved by Nova (1989) is extended to some extent to a general inelastic model.

Key words: existence, geomaterials, localization, non-normality, second-order work, uniqueness

1. Introduction

The behavior of geomaterials (granular materials, soils, and rocks) has some specific properties which create certain mathematical difficulties when constitutive equations modelling this behavior are used in a boundary-value problem. One of these properties can be experienced in everyday life. Walking on a beach after the last wave has filled up all the pores of the sand shows clearly that each footprint drains out the surrounding sand. This means that there is an increase in the pore volume in the surrounding sand, even though it is likely that the mean (effective) pressure increases. This phenomenon clearly shows that granular-material behavior exhibits a coupling between the isotropic volume change and the deviatoric stress. This behavior is rather different from that of metals. This complex behavior explains why so many constitutive equations are elaborated in order to model granular materials and other geomaterials. These constitutive equations are now often based on the well-known framework of classical plasticity (which here means isotropic hardening obeying a normality rule). Many of these depart strongly from classical plasticity theory such as, for instance, multi-mechanism plasticity models [1], bounding-surface plasticity models [2] hypoplasticty models – see [3] for a review – or multi-laminate [4] and microplane [5] models.

These inelastic constitutive equations are used in numerical computations assuming implicitly the well-posedness of the underlying boundary-value problem. It is our opinion that there is a need for knowledge about existence and uniqueness of solutions of boundary-value problems involving such general inelastic models. Except for classical elasto-plasticity constitutive equations (see [6]), for non-associative plasticity (see [7]) and for hypoplasticity (see [8]), there is no simple and general (here general means independent of the geometry and the boundary conditions of the problem) results about existence and uniqueness of solutions of boundary-value problems involving inelastic constitutive equations. A common result can be deduced from the results quoted above. Considering problems solved with the so-called small-strain assumption, we observe that the positiveness of the second-order work (everywhere and for any strain rate) ensures the uniqueness of the solution for classical elastoplastic [9] and hypoplastic [8] models.

On the other hand, it has been proved by Nova for isotropic hardening elastoplastic models [10] – see also [11] – that, if the second-order work can be negative, then it is possible to construct a problem (starting from an homogeneous state) such that, for some specific boundary conditions, the uniqueness of (in this case) the homogeneous solution is lost. This result has been extended to particular cases of hypoplasticity by El Hassan [12]. More recently Niemunis gave a general proof for hypoplastic theories [13]. One of the objectives of this paper is to generalize this result to a wide class of constitutive equations. It is necessary to emphasize that, even if the constitutive equations are rather general, the problem solved is quite particular because it is related to homogeneous states only. This problem has to be related to what is often called material instability [14,9]. It is necessary to quote here the work of Petryk concerning problems similar to the one addressed here for the multi-mechanism plasticity theory. Within this specific framework his work goes beyond the scope of this paper, since he takes into account geometrical nonlinearities [15].

The second objective is to give a sufficient localization condition – *i.e.*, a condition which implies that all the equations of a Rice localization analysis [16] are fulfilled – for any rate-independent materials and to prove that this condition implies the negativeness of the second-order work.

The paper presentation is as follows. In the first part we present the problem under consideration. In particular, the controllability as defined by Nova is recalled. Next we prove that the loss of positiveness of the second-order work implies loss of controllability. The third part deals with the shear-band analysis. A conjecture is explained in a concluding-remarks section.

The following notations are used: a tensor is denoted by an underlined symbol like $\underline{\sigma}$, the component of a tensor (or vector) is denoted by the name of the tensor (or vector) accompanied by lower indices. Other indices and among them upper indices have specific meanings defined in the text. The summation convention with respect to repeated tensorial indices is used.

2. The problem under consideration

2.1. The basic assumptions

We are dealing with inelastic materials not exhibiting viscous effect in the small-strain range. The constitutive equation can thus be written in rate form as follows:

$$\underline{\dot{\sigma}} = \mathcal{F}(\underline{\dot{\varepsilon}}), \tag{1}$$

where $\underline{\dot{\sigma}}$ is the stress rate, $\underline{\dot{\varepsilon}}$ the strain rate and $\mathcal{F}$ a tensorial function depending on the state of the material. It is assumed first that $\mathcal{F}$ is invertible which means defining $\mathcal{G}$ as the inverse of $\mathcal{F}$ so that:

$$\underline{\dot{\varepsilon}} = \mathcal{G}(\underline{\dot{\sigma}}). \tag{2}$$

If this assumption does not hold, it is clear that controllability is lost. Loss of invertibility means: given $\underline{\dot{\sigma}}$, either $\mathcal{G}(\underline{\dot{\sigma}})$ does not exist or at least there are two different $\underline{\dot{\varepsilon}}$ such that Equation (1) holds. Consequently a boundary-value problem corresponding to the boundary conditions compatible with the given value of $\underline{\dot{\sigma}}$ either has no homogeneous solution or has several solutions. In both cases controllability defined hereafter is lost.

As we are studying non-viscous materials, $\mathcal{F}$ and $\mathcal{G}$ are homogeneous of the degree one with respect to their respective arguments. We will add other assumptions when this is necessary. The quantities $\underline{\dot{\sigma}}$ and $\underline{\dot{\varepsilon}}$ belong to the six-dimensional space of symmetric second-order tensors. In the following it will be useful to define an orthonormal basis for this set which means a set of six second-order tensors denoted in the following by ${}^{i}\underline{e}$ such that

$$\forall i,\ \|{}^{i}\underline{e}\| = 1, \qquad \forall i, j,\ \ i \neq j,\ \ {}^{i}\underline{e} \cdot {}^{j}\underline{e} = 0, \tag{3}$$

where $\cdot$ denotes the usual scalar product of two second-order tensors (for instance the scalar product $\underline{\dot{\sigma}} \cdot \underline{\dot{\varepsilon}} = \underline{\dot{\sigma}}_{ij} \underline{\dot{\varepsilon}}_{ij}$ defines the second-order work).

2.2. Controllability

In his paper Nova [10] defined controllability as follows. First he noticed that in some experiments, like the classical triaxial tests, some components of the strain and the other components of the stress are prescribed. He pointed out the practical importance of being able to perform such a test, *i.e.*, to get one (existence) and only one (uniqueness) response for such a test. He noticed then, that in some tests like the undrained ones, a linear combination of the classical strain components is prescribed. Generalizing this remark he defined a new set of strain (and stress) variables related to the strain (or stress) components via a product with a non-singular matrix the inverse of which is equal to its transpose (which means orthonormal change of basis). He proceeded by defining controllability as the ability of a material (or a model) to provide one and only one (existence and uniqueness) response to any loading path for which some strain components (in this new basis) and the other stress components are prescribed.

Finally, for a constitutive equation, controllability in other words is, existence and uniqueness of the solution of the following problem. In a given orthonormal basis some components of the strain rate and the other components of the stress rate are prescribed, and we try to solve the corresponding boundary-value problem. If there exists one and only one solution for the non-prescribed components (of the stress and the strain rates) satisfying the constitutive equation (1) or (2), then the model is said to be controllable. Otherwise it is not controllable.

3. Consequences of the non-positiveness of the second-order work

3.1. The non-positiveness of the second-order work implies non-uniqueness

Rephrasing the problem as in the previous section allows us to prove that, if the second-order work is equal to zero for some strain direction, then the constitutive equation is non-controllable. As already seen in Section 2.1, it is assumed that the constitutive Equation (1) is invertible or else the model is clearly non-controllable. If the second-order work can be equal to zero, there exists some strain rate denoted by $\underline{\dot{\varepsilon}}^0 \neq 0$ such that:

$$\text{if} \quad \underline{\dot{\sigma}}^0 = \mathcal{F}(\underline{\dot{\varepsilon}}^0), \qquad \text{then} \quad \underline{\dot{\sigma}}^0 \cdot \underline{\dot{\varepsilon}}^0 = 0. \tag{4}$$

This means that the second-order work is not strictly positive. Now we choose the orthonormal basis such that:

$$ {}^{1}\underline{e} = \frac{\dot{\underline{\varepsilon}}^{0}}{\|\dot{\underline{\varepsilon}}^{0}\|} \tag{5} $$

and

$$ {}^{2}\underline{e} = \frac{\dot{\underline{\sigma}}^{0}}{\|\dot{\underline{\sigma}}^{0}\|}. \tag{6} $$

Denoting in the following by ${}^{i}a$ and ${}^{i}b$ the components of $\dot{\underline{\varepsilon}}$ and $\dot{\underline{\sigma}}$, respectively, we have

$$ \dot{\underline{\varepsilon}} = {}^{1}a\,{}^{1}\underline{e} + {}^{2}a\,{}^{2}\underline{e} + {}^{3}a\,{}^{3}\underline{e} + {}^{4}a\,{}^{4}\underline{e} + {}^{5}a\,{}^{5}\underline{e} + {}^{6}a\,{}^{6}\underline{e} \tag{7} $$

and

$$ \dot{\underline{\sigma}} = {}^{1}b\,{}^{1}\underline{e} + {}^{2}b\,{}^{2}\underline{e} + {}^{3}b\,{}^{3}\underline{e} + {}^{4}b\,{}^{4}\underline{e} + {}^{5}b\,{}^{5}\underline{e} + {}^{6}b\,{}^{6}\underline{e}. \tag{8} $$

For instance, the components of $\dot{\underline{\varepsilon}}^{0}$ are

$$ \dot{\varepsilon}^{0}, 0, 0, 0, 0, 0, \tag{9} $$

defining $\dot{\varepsilon}^{0}$ and the components of $\dot{\underline{\sigma}}^{0}$ are

$$ 0, \dot{\sigma}^{0}, 0, 0, 0, 0, \tag{10} $$

defining $\dot{\sigma}^{0}$.

Controllability means that for some prescribed components ${}^{i}a$, and the prescribed complementary components ${}^{j}b$ (which then define a mixed loading path), it is possible to find one and only one set of the non-prescribed values of ${}^{i}a$ and ${}^{j}b$, such that the corresponding strain rate and stress rate meet the constitutive equation (1) or (2). Now let us consider the following problem: ${}^{1}b$, ${}^{2}a$, ${}^{3}a$, ${}^{4}a$, ${}^{5}a$ and ${}^{6}a$ are prescribed equal to zero, that is,

$$ {}^{1}b = {}^{2}a = {}^{3}a = {}^{4}a = {}^{5}a = {}^{6}a = 0. \tag{11} $$

Clearly a solution is given by

$$ {}^{1}a = 0, \quad {}^{2}b = 0, \quad {}^{3}b = 0, \quad {}^{4}b = 0, \quad {}^{5}b = 0, \quad {}^{6}b = 0. \tag{12} $$

However, another solution is:

$$ {}^{1}a = \dot{\varepsilon}^{0}, \quad {}^{2}b = \dot{\sigma}^{0}, \quad {}^{3}b = 0, \quad {}^{4}b = 0, \quad {}^{5}b = 0, \quad {}^{6}b = 0. \tag{13} $$

Moreover, due to the positive homogeneity of degree one of the constitutive equation, other solutions are

$$ {}^{1}a = \lambda\dot{\varepsilon}^{0}, \quad {}^{2}b = \lambda\dot{\sigma}^{0}, \quad {}^{3}b = 0, \quad {}^{4}b = 0, \quad {}^{5}b = 0, \quad {}^{6}b = 0, \tag{14} $$

where λ is any positive number. For the same boundary conditions, the ones corresponding to the prescribed values defined in Equation (11), there exists not only the solution given by Equation (12), but also all the solutions given by Equation (14). Clearly, first uniqueness and then controllability is lost.

3.2. The non-positiveness of the second-order work implies either non-existence or *instability* of the solutions

In the previous section, we proved that uniqueness is lost which is sufficient to prove loss of controllability. In the current section, adding some assumptions about the differentiability of the constitutive equation will allow us to prove moreover, either the loss of existence or the fact that two closed prescribed inputs give two rather different solutions. This means in the latter case discontinuity of the (mixed) response of the constitutive equation with respect to the (mixed) input loading conditions. In some sense this property is more important. In numerical computations, it is not so easy to detect non-uniqueness for a fully nonlinear problem like the one we are studying here; see, however, the algorithm proposed by Chambon [17]. On the contrary, it is easy to detect the non-existence or *instability* of the solution which is often related to non-convergence when a full Newton-Raphson method is used to solve accurately the nonlinear boundary-value problem.

3.2.1. *New assumptions*

We assume now that the functions $\mathcal{F}$ and $\mathcal{G}$ are continuous and differentiable, except in the vicinity of the null tensor. This is not so strong a restriction. However, the flow theory of plasticity does not meet this condition for a strain (or stress) rate direction corresponding to neutral loading. On the other hand, if $\underline{\dot{\varepsilon}}^0$ is not within this (very) restricted set of strain rates, the following applies also to elasto-plasticity. Towards the end of Section 3.2.3, we will extend the proof to constitutive equations involving several mechanisms (several means here two or more than two), but for the beginning we assume the following. Constitutive Equation (1) or (2) can be rewritten in the following form.

$$ {}^{i}b = {}^{i}\varphi({}^{j}a), \tag{15} $$

where $i, j \in \{1, 2, 3, 4, 5, 6\}$ and ${}^{i}\varphi$ are functions that are positively homogeneous of degree one, continuous and differentiable, except for ${}^{j}a = 0$, $\forall j$. Let us write

$$ {}^{ij}A = \frac{\partial\, {}^{i}\varphi}{\partial\, {}^{j}a}, \tag{16} $$

which are only defined in the vicinity of a given strain direction.

3.2.2. *Proof of the property*

We are now looking at the constitutive equation in the vicinity of $\underline{\dot{\varepsilon}}^0$.

A variation of ${}^{1}a = \dot{\varepsilon}^0$ means only a variation of the magnitude of $\underline{\dot{\varepsilon}}^0$. Since the constitutive equation is positively homogeneous and since $\underline{\dot{\sigma}}^0 = \mathcal{F}(\underline{\dot{\varepsilon}}^0)$, a variation of the magnitude of $\underline{\dot{\varepsilon}}^0$ implies only a variation of the magnitude of $\underline{\dot{\sigma}}^0$. This implies that:

$$ {}^{i1}A(\underline{\dot{\varepsilon}}^0) = 0 \quad \forall i \neq 2 \tag{17} $$

and

$$ {}^{21}A(\underline{\dot{\varepsilon}}^0) = \frac{\dot{\sigma}^0}{\dot{\varepsilon}^0}. \tag{18} $$

Let us now look for a solution of the following problem where

$$ {}^{1}b = \alpha \quad {}^{2}a = 0 \quad {}^{3}a = \beta \quad {}^{4}a = {}^{5}a = {}^{6}a = 0 \tag{19} $$

are prescribed. The parameters α and β are assumed to be small with respect to $\dot{\varepsilon}^0$. When $\alpha=\beta=0$, we have the following solutions

$$\forall\lambda,\quad {}^1a=\lambda\dot{\varepsilon}^0\quad {}^2b=\lambda\dot{\sigma}^0\quad {}^3b=0\quad {}^4b=0\quad {}^5b=0\quad {}^6b=0. \tag{20}$$

Thus, if $0\neq\alpha\neq\beta\neq 0$, and if we are looking for a solution close to a previous one, such that $\lambda\neq 0$, then we can use the derivatives of the constitutive equation defined in (16) in the vicinity of $\underline{\dot{\varepsilon}}^0$ or in the vicinity of $\lambda\underline{\dot{\varepsilon}}^0$ (Owing to positive homogeneity these derivatives are the same). This implies that necessarily

$${}^1a=\lambda\dot{\varepsilon}^0+\gamma \tag{21}$$

with γ being of the same order as α and β. So finally this implies:

$${}^1b=\alpha={}^{11}A\gamma+{}^{13}A\beta={}^{13}A\beta. \tag{22}$$

Then $\alpha={}^{13}A\beta$, which contradicts the fact that α and β are chosen independently, and so generally a solution of our problem does not exist in the vicinity of the direction of $\underline{\dot{\varepsilon}}^0$.

Let us summarize this result. There is no solution for the problem just defined in the vicinity of the direction of $\underline{\dot{\varepsilon}}^0$ $\forall\lambda\neq 0$. Thus, either there is no solution at all (non existence) or, if there is a solution, it will not be in the vicinity of the direction of $\underline{\dot{\varepsilon}}^0$. This means in this case that for two close inputs given, respectively, by Equations (11) and (19) the corresponding solutions are not close to each other, which concludes the proof.

3.2.3. *Extension of the proof*

Let us now relax the assumptions of the beginning of Section 3.2.1 in order to apply our result to multi-mechanism plasticity. We assume that for some strain direction it is possible to define a finite number (say n) of zones $Z^k, k\in 1,\dots,n$ in the strain space. When $\underline{\dot{\varepsilon}}^0$ belongs to the boundary of these zones and for every zone Z^k, we can define

$${}^{ij}A^k=\frac{\partial\,{}^i\varphi}{\partial\,{}^ja}, \tag{23}$$

which are all defined in the vicinity of $\underline{\dot{\varepsilon}}^0$.

In this case, according to Section 3.2.2, if we are looking for a solution close to $\underline{\dot{\varepsilon}}^0$, we get $\alpha={}^{13}A^k\beta$ for at least one k, which still contradicts the fact that α and β are chosen independently. So it is possible to generalize the previous proof.

4. Shear-band analysis

4.1. The problem of shear banding

It is often claimed that shear banding corresponds to a zero value of the determinant of the acoustic tensor. The problem is that such a tensor can be defined using a dynamic analysis only for incrementally linear models. Generally speaking, a shear band can be generated if the following conditions hold [16]. We consider the problem of an initially homogeneous solid strained up to the current state. It is then submitted to a load rate on a straight loading path. A solution of the resulting rate-equilibrium problem corresponds to an homogeneous strain rate denoted by $\underline{\dot{\varepsilon}}^{\text{out}}$. Another solution involving the existence of a shear band is considered. It is assumed that the strain rate is equal to $\underline{\dot{\varepsilon}}^{\text{out}}$ outside a shear band and equal to

$$\underline{\dot{\varepsilon}}^{\text{in}}=\underline{\dot{\varepsilon}}^{\text{out}}+g\otimes n \tag{24}$$

inside the shear band. The vector n is normal to the shear band and g is some vector. Let $\dot{\underline{\sigma}}^{\text{in}}$ be the Cauchy stress rate with respect to a fixed frame inside the shear band and $\dot{\underline{\sigma}}^{\text{out}}$ outside. Along the boundaries of the band, equilibrium equations in a rate form can be written:

$$\dot{\underline{\sigma}}^{\text{in}} \cdot n = \dot{\underline{\sigma}}^{\text{out}} \cdot n. \tag{25}$$

Moreover the constitutive Equation (1) or (2) has to be satisfied inside and outside the band.

4.2. General criterion and consequences

For any constitutive equation shear bands are possible if there exist some n and some g such that

$$\mathcal{F}(\frac{1}{2}(g \otimes n + n \otimes g)) \cdot n = 0, \tag{26}$$

which means that

$$\dot{\underline{\sigma}} \cdot n = 0, \tag{27}$$

for the stress rate corresponding to the strain rate $\frac{1}{2}(g \otimes n + n \otimes g)$. In Equations (26) and (27), 0 like n or g is a vector.

It is easy to prove that if it is assumed that $\dot{\underline{\varepsilon}}^{\text{out}} = 0$, which means physically negligible with respect to $g \otimes n$, then Equation (26) implies Equation (24) and Equation (25). Then we have obtained a sufficient localization criterion available for any constitutive equation. Moreover, as pointed out by Nova [18] for elasto-plastic models and by Chambon [19] for hypoplastic models, shear-band localization implies that the corresponding second-order work is equal to zero, since by writing Equation (27) in component form, we have

$$\dot{\underline{\sigma}}_{ij} n_j = 0, \tag{28}$$

which implies

$$\dot{\underline{\sigma}}_{ij} n_j g_i = 0, \tag{29}$$

which is the second-order work written for the strain rate $\frac{1}{2}(g \otimes n + n \otimes g)$, since $\underline{\sigma}_{ij}$ is symmetric.

5. Concluding remarks

In Section 3, we studied homogeneous problem. This means that it is possible to put on an homogeneous sample, with boundary conditions corresponding to the prescribed components (in the basis defined above) of the strain or of the stress. Physically, this implies only the use of control devices and this can be actually encountered (for instance in undrained tests), as pointed out by Nova [18]. But we have to keep in mind that this kind of boundary conditions, as seen at the end of Section 2.2, linking different components, is not usually taken into account in classical existence and uniqueness theorem.

However, usually, existence uniqueness and controllability are proved for very particular constitutive equations. Indeed, the theorems proved here are almost independent of the constitutive equation.

Since we have proved recently that, for materials not obeying the normality rule, the second-order work can become negative strictly inside the limit surface [20], it is then possible to lose the well-posedness of the rate boundary-value problem far in advance of what is classically seen as rupture. Let us emphasize that it is possible to define normality in a general manner without reference to a particular constitutive equation; see [20, Section 4.3.2]. So the previous proposition is true irrespective of the constitutive equation.

Al these discussions are very important for geomaterials which are well known not to obey normality rules, again without any reference to a particular law.

Clearly the loss of controllability of some homogeneous problem, when the second-order work can be negative, does not imply that uniqueness (or existence) is lost if the second-order work can be equal to zero or even negative in a subset of a studied domain. This statement is sustained by the stability study of Dascalu *et al.* [21] which proved for a linear system that the system can remain stable (which implies existence and uniqueness of the static system) if a limited fault undergoes softening provided that the corresponding (negative) softening modulus has a small absolute value.

We have studied only a rate problem here. It would be desirable to deal with the more interesting initial-boundary-value problem; however, the latter problem is more difficult to tackle.

Finally let us conclude by stating a conjecture. For a reasonable constitutive equation (continuous and smooth enough), when the second order is positive everywhere, and for any strain rate, then, independently of the boundary conditions (with the same restrictions as for classical elastic computations) and of the shape of the studied domain, the small-strain-rate boundary-value problem is well-posed. If, on the contrary, for some points of the domain and some strain directions, the second-order work is negative, it is possible that the corresponding small-strain-rate boundary-value problem is ill-posed.

References

1. W.T. Koiter, General theorems for elstic plastic solids. In: I.N. Sneddon, R. Hill (eds.), *Progress in Solid Mechnics* 1 Amsterdam: North Holland Publishing Comp. (1960) pp. 165–221.
2. Y. Dafalias and I. Popov, Plastic internal variables formalism of cyclic plasticity. *J. Appl. Mech.* 98 (1976) 645–650.
3. C. Tamagnini, G. Viggiani and R. Chambon, A review of two different approaches to hypoplasticity. In: D. Kolymbas (ed), *Constitutive Modelling of Granular Materials including Developments and Perspectives of Hypoplasticity*. Berlin: Springer (1999) pp. 107–145.
4. O.C. Zienkiewicz and G.N. Pande, Time-dependent multi-laminate model of rocks. A numerical study of deformation and failure of rock masses. *Int. Num. Analyt. Methods Geomech.* 1 (1977) 219–247.
5. M. Jirasek and Z.P. Bazant, *Inelastic Analysis of Structures*. Chichester: John Wiley and sons (2002) 758 pp.
6. R. Hill, Aspect of invariance in solids mechanics. *Adv. Appl. Mech.* 18 (1978) 1–75.
7. B. Raniecki and O.T. Bruhns, Bounds to bifurcation stresses in solids with non associated plastic flow rule at finite strain. *J. Mech. Phys. Solids* 29 (1981) 153–172.
8. R. Chambon and D. Caillerie, Existence and uniqueness theorems for boundary-value problems involving incrementally non linear models. *Int. J. Solids Struct.* 36 (1999) 5089–5099.
9. D. Bigoni, Bifurcation and instability on non associative elastoplastic solids. In: H. Petryk (ed.), *CISM Material Instabilities in Elastic and Plastic Solids.* Wien, New York: Springer (1999) pp. 1–52.
10. R. Nova, Controllability of the incremental response of soil specimens subjected to arbitrary loading programmes. *J. Mech. Behav. Materials* 5 (1994) 221–243.
11. S. Imposimato and R. Nova, An investigation on the uniqueness of the incremental response of elstoplastic models for virgin sand. *Mech. Cohesive Frictl. Material* 3 (1998) 65–87.
12. N. El Hassan, *Modélisation théorique et numérique de la localisation de la déformation dans les géomatériaux*. PhD thesis, Université Joseph Fourier Grenoble France (1997) 345 pp.
13. A. Niemunis, *Extended Hypoplasticiy for Soils*. Habilitation thesis, Bochum, Germany (2002) 191 pp.
14. H. Petryk, Theory of material instability in incrementally nonlinear plasticity. In: H. Petryk (ed.), *CISM Material Instabilities in Elastic and Plastic Solids*. Wien, New York: Springer (1999) pp. 261–331.

15. H. Petryk, General conditions for uniqueness in materials with multiple mechanisms in inelastic deformation. *J. Mech. Phys. Solids* 48 (2000) 367–396.
16. J. Rice, The localization of plastic deformation. In: W.D. Koiter (ed.), *Proceedings of the International Congress of Theoretical and Applied Mechanics*. Amsterdam: North Holland Publishing Comp. (1976) pp. 207–220.
17. R. Chambon, S. Crochepeyre and R. Charlier, An algorthm and a method to search bifurcation points in non linear problems. *Int. J. Num. Methods Engng.* 51 (2001) 315–332.
18. R. Nova, Liquefaction, stability, bifurcation, of soil via strainhardening plasticity. In: Z. Sikora (ed.), *Proceedings of the International Workshop on Bifurcation and Localization in Soil and Rocks*. Gdansk (1989) pp. 117–131.
19. R. Chambon and S. Crochepeyre, Daphnis a new model for the description of post-localization behavior: application to sands. *Mech. Cohesive Frictl. Material.* 3 (1998) 127–153.
20. R. Chambon and V. Roger, Mohr-Culomb mini CLoE model: uniqueness and localization studies, links with normality rule. *Int. J. Num. Analyt. Methods Geomech.* 27 (2003) 49–68.
21. C. Dascalu, I. R. Ionescu and M. Campillo, Fault finitness and initiation of dynamic shear instability. *Earth Planet. Sci. Lett.* 177 (2000) 163–176.